# THE "VALUE OF THE GAME" AS A TOOL IN THEORETICAL ECONOMICS

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## INTRODUCTION

The "value" is a solution concept for cooperative n-person games that gives for each player, an unbiased measure of his expected marginal worth to a coalition formed at random. In this paper, several games based on very simple economic models dealing with ownership, production, and exchange will be formulated and solved for their values. These examples have been selected (more for their methodology than their economics) from a dozen or so comparable economic games treated in recent papers by Martin Shubik and the present author.**

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**See references [8] and [9] at the end of this paper.

DEFINITION OF THE VALUE

The value of the game is most easily defined when we can adequately represent the game by a numerical "characteristic function", $v(S)$, which states how much each coalition $S$ of players can assuredly win, regardless of the other players' actions. Such a representation is not always an adequate description of the game, at least for the purposes of value theory. In particular, we must stipulate that the worst threats that might be made against a coalition have a negligible cost to the threateners, else the numbers $v(S)$ will be too pessimistic. Moreover, the use of single number $v(S)$ to describe the worth of a coalition implies that the players measure their utility on the same scale, and have the ability to transfer it freely within the coalition. Without these two assumptions, concerning costly threats and the existence of a "money", value theory becomes more difficult, both conceptually and computationally. (See the final section of this paper.) Fortunately, many economic situations meet these conditions, at least to a first approximation.

Given the characteristic function, the following probability model puts the definition of the value in a convenient form for calculation. Let the players be arranged in a random order, with all orderings equally likely, and let $P_i$ be the random variable denoting the set of predecessors of player $i$. Then the value to $i$ is

$$\phi_i = E\{v(P_i \cup \{i\}) - v(P_i)\}.$$

It has been shown (see [4]) that this value is the unique single-outcome solution of a characteristic-function game that satisfies certain simple postulates of symmetry, efficiency, and additivity. In particular, the value is Pareto optimal:

$$\sum_I \phi_i = v(I),$$

if I denotes the all-player set.

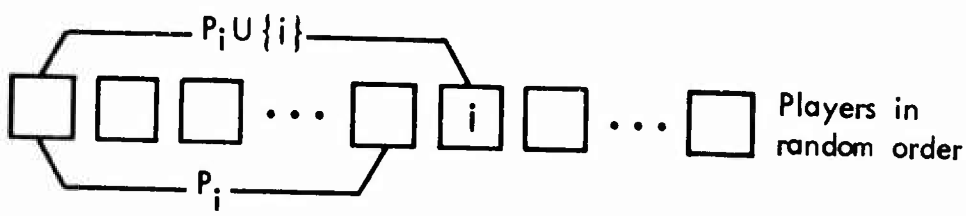


Fig. 1—Random coalition-building

MODEL 1: The landlord and the peasants*

Let the first player own a cornfield, and let players 2, 3, ..., n be peasants, owning no land. Let the money value of the crop be some function of the labor input:

$$p = f(\ell), \qquad \ell = \text{no. of laborers.}$$

This may be an S-shaped curve, as in the illustrations, or more generally any monotonic curve increasing from the origin. The characteristic function (assuming the landlord does not work) is:

$$v(S) = \begin{cases} f(|S| - 1) & \dots \text{ if } 1 \in S \\ 0 & \dots \text{ if } 1 \notin S, \end{cases}$$

where "$|S|$" denotes the number of members of S.

The values of this game are easy to determine, because of the symmetry. Merely insert the landlord (player 1) in each of the n possible positions in a random ordering of players. In position k, his marginal worth is exactly f(k–1). Thus,

$$\phi_1 = \frac{1}{k} \sum_{k=1}^{n} f(k-1) \approx \frac{1}{n} \int_0^n f(x)dx.$$

Since the total value of the game is f(n–1), and since the other players are all alike, we have

*See [8], Sec. V.

$$\phi_2 = \phi_3 = \dots = \phi_n = \frac{f(n-1) - \phi_1}{n-1} .$$

The apportionment of the crop represented by the value solution has a simple geometric representation (see Figure 2). The landlord's share is proportional to the area beneath the production curve; the peasants' share is proportional to the area to the left of the curve.

This "fair division" solution may be compared with the "pure competition" solution, in which each laborer receives a wage equal to his marginal productivity at full production (see Figure 3). This wage may be greater or less than the value of the game, depending on the shape of the production curve.

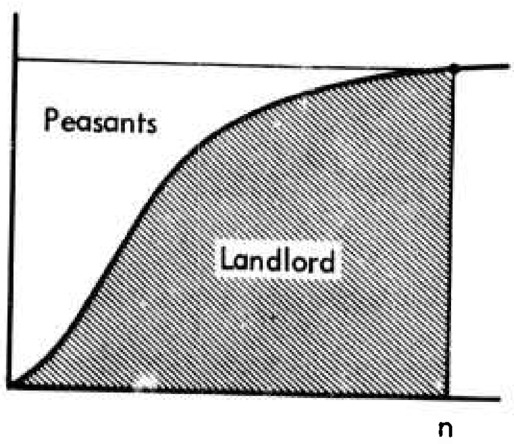


Fig. 2—Apportionment of the value

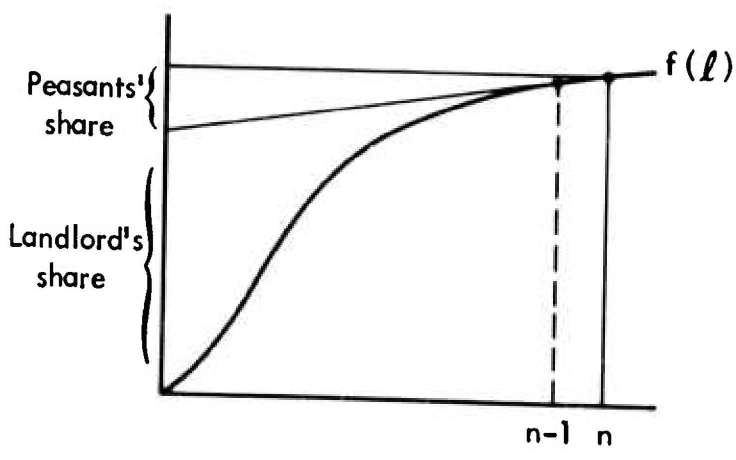


Fig. 3—The competitive solution

MODEL 2: The small landowners*

Now divide the cornfield of the previous example into $n$ individually-owned plots, of sizes $c_1 + c_2 + \cdots + c_n = C$. Suppose that for some reason there is a decrease in efficiency when plots smaller than $C$ are cultivated. In fact, let the production function have the form

$$p = F(\ell, c) = \frac{c}{C} f(\ell),$$

with $f$ as before. For further variety, let the players have possibly different amounts of labor: $\ell_1 + \ell_2 + \cdots \ell_n = L$.

The form of $F$ ensures that any coalition will want to cultivate its plots as a unit.** Thus, the total value of the game is $F(L, C) = f(L)$, and the characteristic function is given by

$$v(S) = F\Big(\sum_S \ell_i, \sum_S c_i\Big) = \frac{1}{C} \sum_S c_i \; f\Big(\sum_S \ell_i\Big).$$

In such a game, if each player is already small compared to the whole, it can be shown that the values do not change significantly if the players are broken up into smaller players, each set of "fragments" owning the same totals of land and labor as the corresponding original player. The

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*See [8], Sec. VI.

**This can be seen from the inequality $c'f(\ell') + c''f(\ell'') \leq c'f(\ell' + \ell'') + c''f(\ell' + \ell'') = (c' + c'')f(\ell' + \ell'')$, which depends only on the monotonicity of $f$.

limiting case proves easy to handle; it is a so-called "measure game" on a continuous infinity of individually insignificant players. In this kind of game, the all-player set is a measure space, P, and the coalitions are the measurable subsets of P. The characteristic function can be expressed as follows:

$$v(S) = F(\lambda(S), \gamma(S)),$$

where $\lambda$ and $\gamma$ are atomless measures on P with $\lambda(P) = L$, $\gamma(P) = C$.

Since individual players have value zero, the value solution must also be expressed as a measure on P. In fact, there is a rather remarkable explicit formula for the value of this game,* namely:

$$\phi(S) = \int_0^1 [\lambda(S)F_\ell(tL, tC) + \gamma(S)F_c(tL, tC)]dt,$$

where $F_\ell$ and $F_c$ are the partial derivatives of F. The only condition is that F be continuously differentiable in a neighborhood of the diagonal $\{(tL, tC) \mid 0 \leq t \leq 1\}$ — an assumption we are quite willing to make. Observe that $\phi$, the value measure, turns out to be linear combination of the measures $\lambda$ and $\gamma$.

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*See [1]. The formula generalizes in the obvious way to functions of any finite number of measures.

To return to the original problem, we must identify each player $i$ of the original game with a coalition $S_i$ in the measure game such that $\lambda(S_i) = \ell_i$ and $\gamma(S_i) = c_i$. This yields the approximation

$$\phi_i \approx \left[A \frac{c_i}{C} + (1-A) \frac{\ell_i}{L}\right] f(L),$$

where the constant $A$ is the relative area beneath the production curve $f$:

$$A = \frac{\int_0^L f(\ell)d\ell}{Lf(L)} .$$

(This constant figured in the previous model. In fact, if we give all the land to one player, the present approximation agrees with what we found in the previous example, despite the presence of a "large" player which might have undermined the validity of the approximation.)

Summing up: in this simple production model, the "value" apportionment of the output is a compromise between sharing according to the capital contribution and sharing according to the labor contribution. The relative weight given to these two input factors depends on the integral of the production function $f$, in an intuitively satisfactory way. For example, a small area to the left of the curve (A close to 1) means that labor is unlikely to be in short supply, even if a subcoalition goes into business for itself, and a

man's value depends chiefly on the land he can provide. Conversely, a small area under the curve (A close to zero) means that labor is generally the critical input, and the value solution gives little weight to the distribution of land ownership.

MODEL 3: Trading in complementary goods.*

Let there be two groups of players, R and L. Each member of R starts with one right glove; each member of L with one left glove. The players may trade gloves, or buy and sell them for money, without restriction. At the end of the game, an assembled pair of gloves can be cashed in for $1 a pair, but odd gloves are worthless.

The characteristic function is given by

$$v(S) = \min(|S \cap R|, |S \cap L|).$$

Let $r = |R|$, $\ell = |L|$. Then the total value of the game is just $\min(r, \ell)$ dollars.

We shall calculate the values to the players as a function of the parameters $r$ and $\ell$. Let $\Phi(r, \ell)$ denote the sum of the values to the members of R, and assume that $r > \ell > 0$. The boundary conditions for this case are

$$\Phi(r, 0) = 0 \quad \text{and} \quad \Phi(r, r) = r/2.$$

If we order the players at random, and if we consider separately those orderings that end with a member of R (prob. $r/(r+\ell)$) and those that end with a member of L (prob. $\ell/(r+\ell)$), we obtain a simple difference equation:

*See [9], Sec. 2; also [5].

$$\Phi(r, \ell) = \frac{r}{r+\ell}\,\Phi(r-1, \ell) + \frac{\ell}{r+\ell}\,\Phi(r, \ell-1).$$

With the stated boundary conditions this has the unique solution

$$\Phi(r, \ell) = \frac{r}{2} - \frac{r-\ell}{2}\sum_{k=0}^{\ell} \frac{r!\,\ell!}{(r+k)!\,(\ell-k)!},$$

as the reader may verify directly. This amount must be divided equally among the members of R, by symmetry, and the balance must be divided equally among the members of L. Hence the value solution, for $r \geq \ell$, is

$$\phi_i = \frac{1}{2} - \frac{r-\ell}{2r}\sum_{k=0}^{\ell} \frac{r!\,\ell!}{(r+k)!\,(\ell-k)!}, \qquad i \in R$$

$$\phi_j = \frac{1}{2} + \frac{r-\ell}{2r}\sum_{k=1}^{\ell} \frac{r!\,\ell!}{(r+k)!\,(\ell-k)!}, \qquad j \in L.$$

The case $r \leq \ell$ is exactly similar. Table 1 gives an idea of how the values behave for small $r$ and $\ell$.

The value solution definitely favors the "short" side of the market, individually and collectively. For example, if $\ell < r$, the members of L, with less than half of the population, get more than half the profit. On the other hand, the "long" side of the market is not totally defeated, as it would be under a competitive price system where the price of the good in oversupply would necessarily be zero.

Table 1

VALUE TO A MEMBER OF R

| r \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | .500 | .667 | .750 | .800 | .833 | .857 | .875 | .889 |
| 2 | 0 | .167 | .500 | .650 | .733 | .786 | .822 | .847 | .867 |
| 3 | 0 | .083 | .233 | .500 | .638 | .720 | .774 | .811 | .838 |
| 4 | 0 | .050 | .133 | .272 | .500 | .629 | .710 | .764 | .802 |
| 5 | 0 | .033 | .086 | .168 | .297 | .500 | .622 | .701 | .755 |
| 6 | 0 | .024 | .060 | .113 | .194 | .315 | .500 | .616 | .693 |
| 7 | 0 | .018 | .044 | .081 | .135 | .214 | .330 | .500 | .610 |
| 8 | 0 | .014 | .033 | .061 | .099 | .153 | .230 | .341 | .500 |

The value solution gives some credit for the bargaining power of the "long" side, though the credit is small unless $r$ and $\ell$ are almost equal.*

The two graphs in Figure 4 show the effect of varying the ratio of trader types in a market of fixed size. In the second graph, with ten times as many traders, the slope of the curve in the vicinity of the transition case $r = \ell$ is noticeably steeper. In the limit, the curve approaches the $\urcorner\!\llcorner$ -shape associated with the competitive equilibrium.**

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*For example, if $r = \ell + 1$, then the members of R, faced with disaster under pure competition, might select two of their number to withdraw from the market, with or without compensation, and thus turn the tables on L. This behavior would not be Pareto optimal, since only $\ell - 1$ pairs of gloves could be formed, but the threat would be credible enough and might well raise the price of right-handed gloves.

The reader will recognize this as a common price-support tactic in situations where collusion is possible. Of course, the value of the game does not directly consider such details of process, but it does recognize and measure the coalition potentials that make such maneuvers effective.

**Other asymptotic properties are discussed in [9], *loc. cit.*

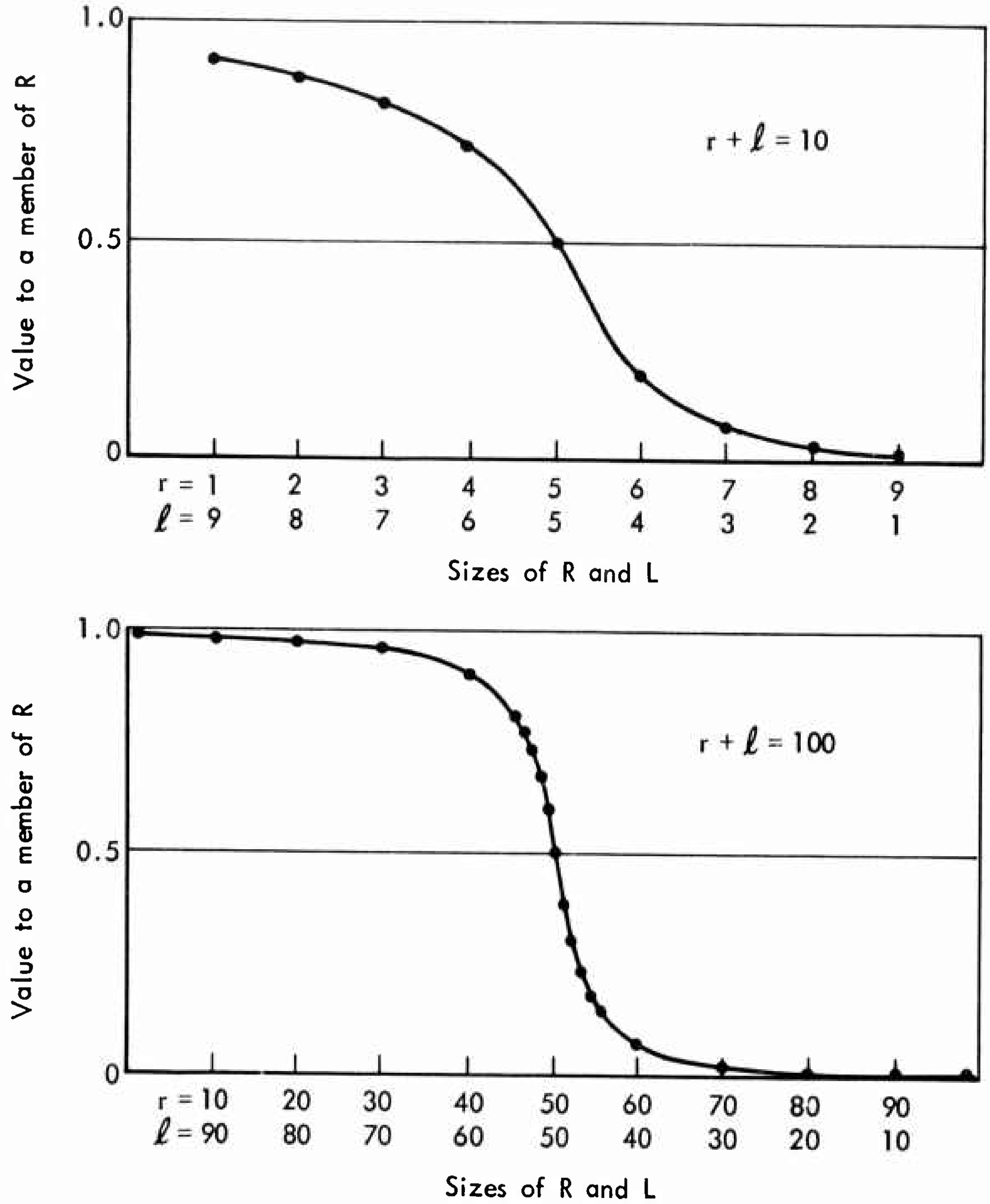


Fig. 4—Value as a function of composition

## MORE GENERAL KINDS OF GAMES

When the strategic possibilities include threats that are costly to the threatener (and hence that should perhaps be discounted in some way), one may employ a modified characteristic function, discovered by Harsanyi, that takes account of this added strategic richness. Given the new function, which is somewhat more trouble to compute, the value calculation proceeds as before.*

When a money or other vehicle for the free transfer of utility does not exist, one may determine a generalized value by introducing hypothetical exchange rates between the players' utilities, in such a way that the transferable-utility value solution can be attained without actually making any transfers. This can always be done, and often in a unique way.**

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*See [2], also Selten's axiomatization [3].

**See [6], [7]. An example, based on the "Edgeworth box" is worked through in detail in [9], Sec. 4.

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